What is dimension?
An invitation to Geometric Measure Theory

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CSplash April 5th

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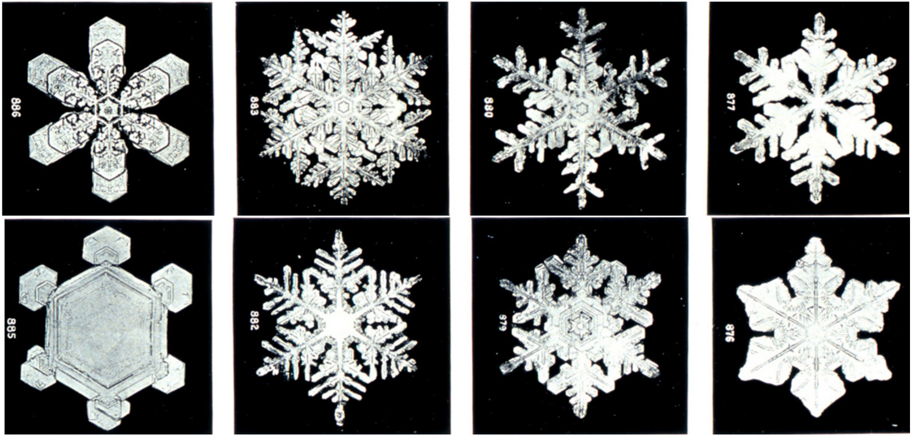
Why study dimension?

Have you ever thought about snowflakes?



Why study dimension?

What about trees?

Why study dimension?

What about the veins on a leaf?

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- Is there a way to measure something about their **structure**?
- Can we find a *Good* measure, to be able to compare a snowflake to a leaf, or to a tree?

In this talk I will show you one way to measure the **dimension** of any snowflake or general shapes!

What is Dimension?

Food for thought

It's easy to say what is the dimension of the shapes below.

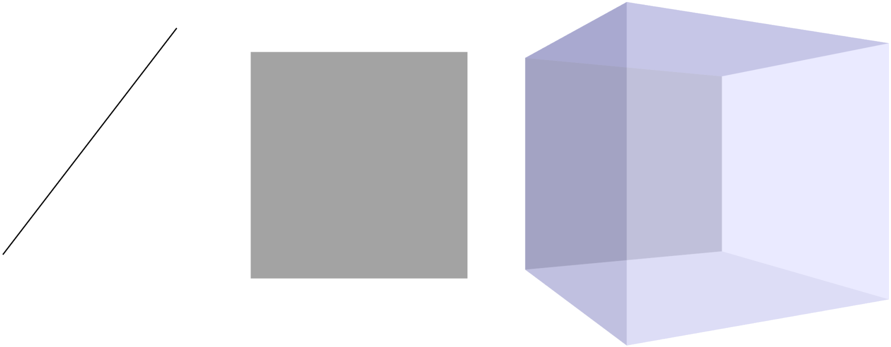

Figure: The line is 1D, the square is 2D and the cube is 3D!

A naive answer

When something has a whole-number dimension, like d, it means that we need d parameters to find any point.

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In a space with d dimensions, you have d degrees of freedom to move!

This intuition works well for integers! But what if you are moving on a snowflake? What about on the branches of a tree? What about the Serpinski triangle? The Koch's snowflake?

Koch Snowflake

The Koch's snowflake looks like the following:

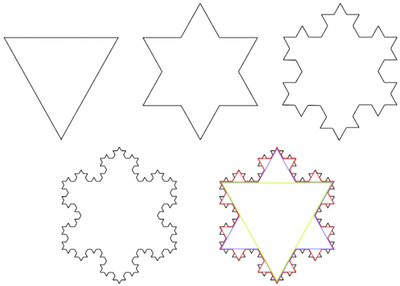
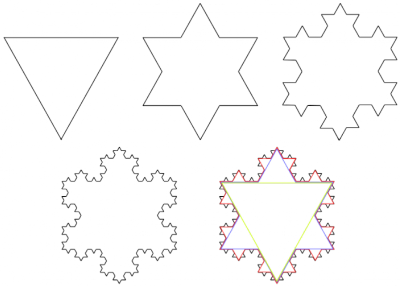


Figure: The process of making a Koch snowflake.

The Koch's snowflake looks like the following:



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At each step the length in multiplied by $4/3$. This says that the length increases exponentially. In other words the 1-dimensional volume (length) of the end object cannot be finite.

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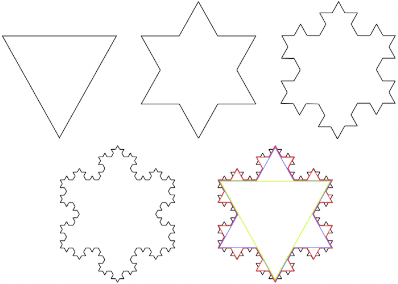

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At each step the length in multiplied by $4/3$. This says that the length increases exponentially. In other words the 1-dimensional volume (length) of the end object cannot be finite.

We need to rethink *What dimension actually means*!

The straight line: covering

Think about the straight line of length 1.

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Think about the straight line of length 1.
How many boxes of size $1/3$ do we need to cover this line?

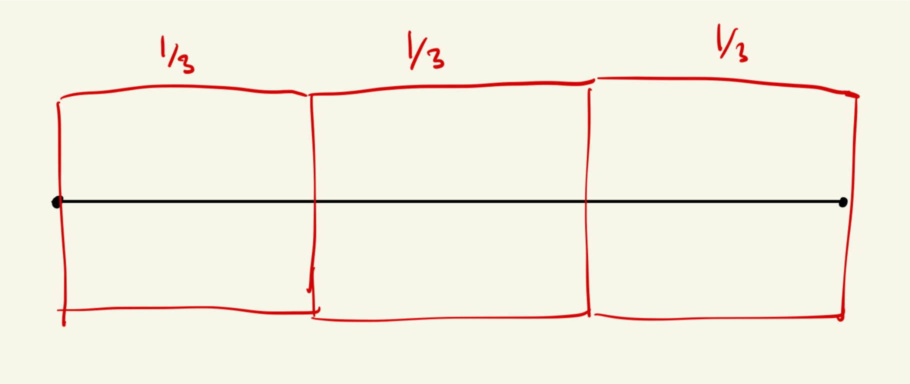

Figure: We need 3^1 boxes of size $1/3$.

How many boxes of size $1/8$ do we need to cover this line?

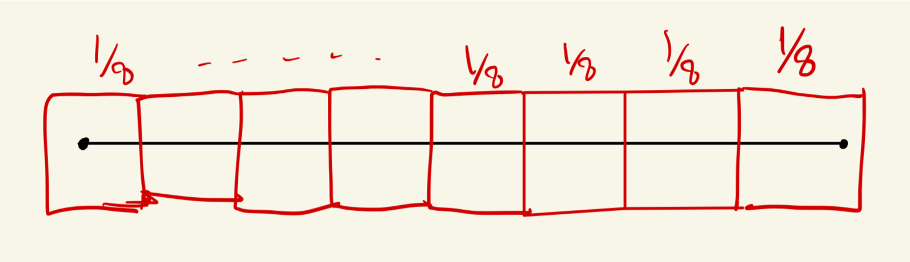

Figure: We need 8^1 boxes of size $1/8$.

As you all guessed: How many boxes of size $1/26$ do we need to cover this line?

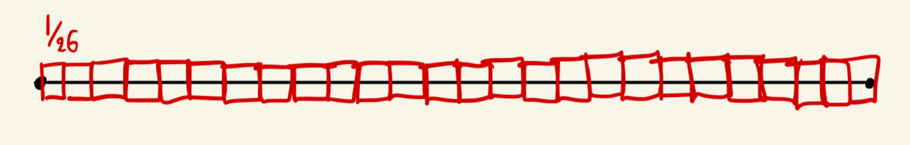

Figure: We need 26^1 boxes of size $1/26$.

The pattern

- We need 3 boxes of size $1/3$.
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For the line of length 1:

We need $\left(\frac{1}{\epsilon}\right)^{1}$ boxes of length ϵ to cover the line of length 1.

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For the line of length 1:

We need $\left(\frac{1}{\epsilon}\right)^{1}$ boxes of length ϵ to cover the line of length 1.

For the rest of the talk we will denote by number of boxes of size ϵ with N_ϵ.

Food for thought 1

What about a curly line?

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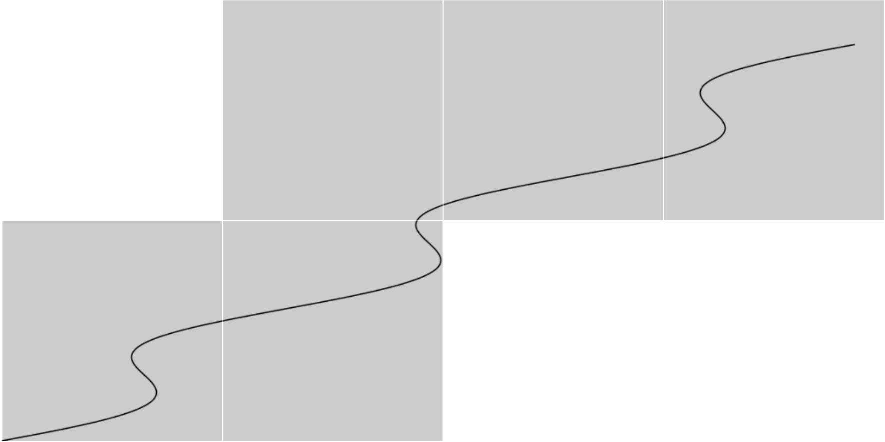

Figure: 5 boxes of size 256 pixels.

Food for thought 1

Finer:

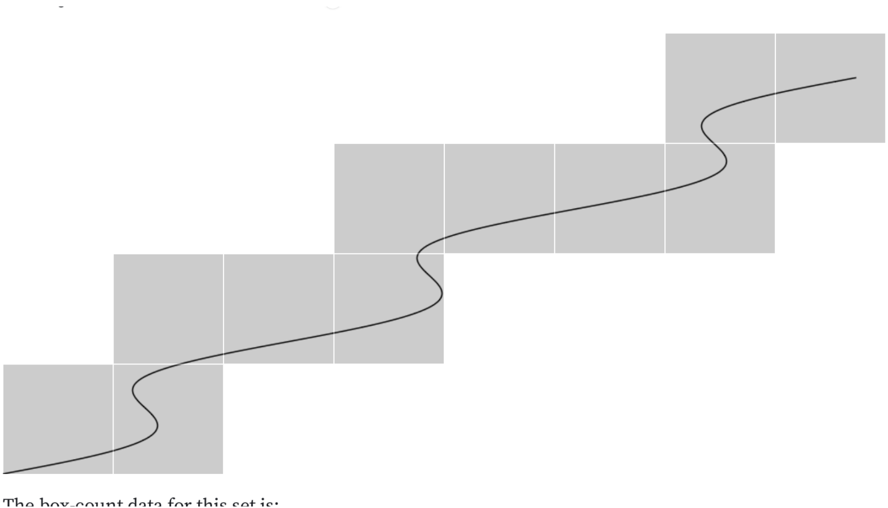

Figure: 11 boxes of size 128 pixels.

Food for thought 1

And finer:

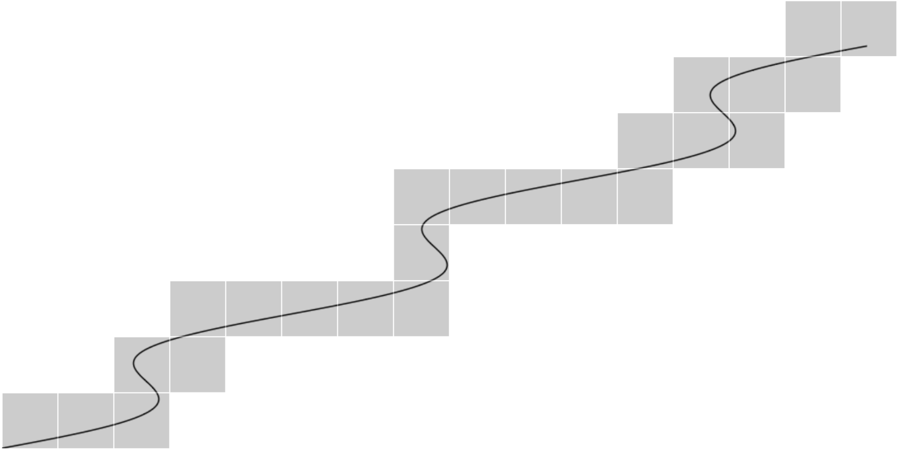

Figure: 24 boxes of size 64 pixels.

Food for thought 1

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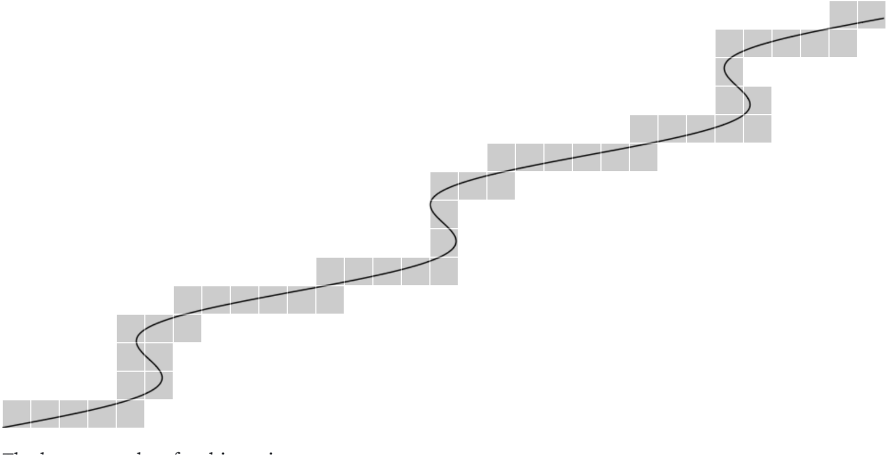

Figure: 49 boxes of size 32 pixels.

Food for thought 1

And finer:

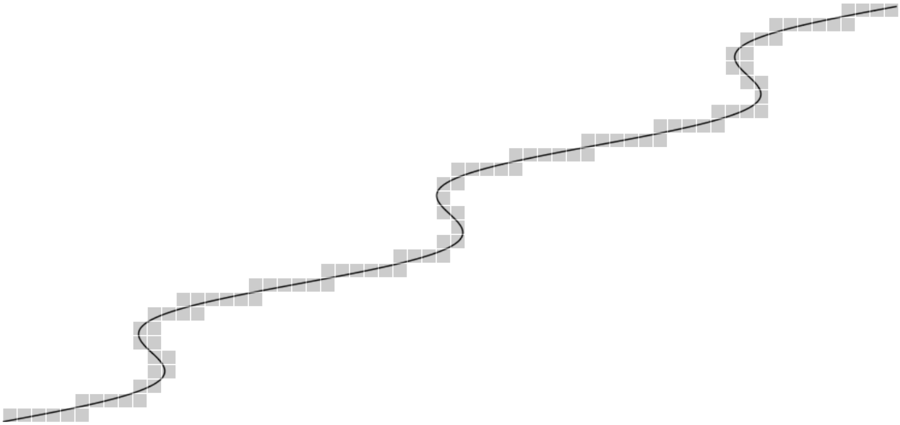

Figure: 101 boxes of size 16 pixels.

Food for thought 1

And finer:

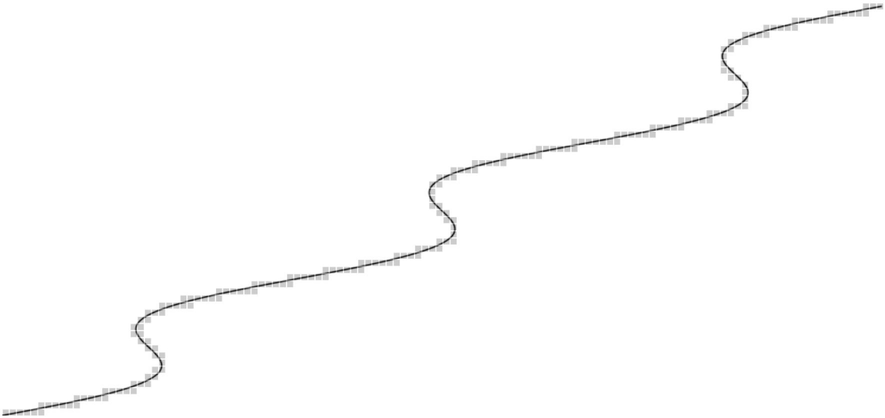

Figure: 201 boxes of size 8 pixels.

Analyzing the numbers:

Let's look again at the numbers:

$$\begin{cases} N_{256} = 5 \\ N_{128} = 11 \\ N_{64} = 24 \\ N_{32} = 49 \\ N_{16} = 101 \\ N_{8} = 201 \end{cases}$$

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$$\begin{cases} N_{256} = 5 \\ N_{128} = 11 \\ N_{64} = 24 \\ N_{32} = 49 \\ N_{16} = 101 \\ N_8 = 201 \end{cases}$$

When we halve the size, the number of squares double. This makes sense!

The Log-Log graph

If we plot $\log(N_\epsilon)$ versus $\log(\epsilon)$ we see this graph:

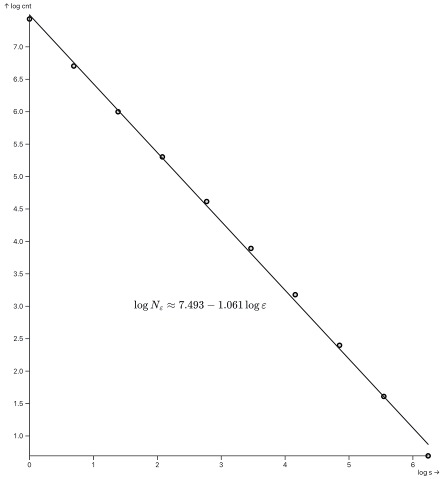

Figure: The slope is one!

The inaccuracy in this estimate is the fact that large scales see less detail.

The linear relation

Not very surprising, because to cover a curve we need:

$$\text{(Length)}.\left(\frac{1}{\epsilon}\right)^{1} \text{ number of boxes of size } \epsilon\,.$$

Taking a logarithm, we see:

$$\log(N_\epsilon) = \log(\text{Length}) + (\text{Dimension})\,.\log(1/\epsilon)\,.$$

Using linear regression and data analysis techniques we can calculate this empirically!

Intuition

It looks like dimension could be about the **growth** of number of boxes we need to cover versus their **size**.

Food for thought 2

Let's test this out for a square (easy thought experiment).

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Like before we start covering:

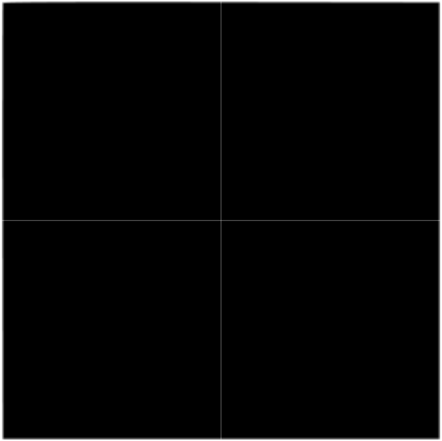

Figure: 4 squares of side length $1/2$.

Food for thought 2

Like before we start covering:

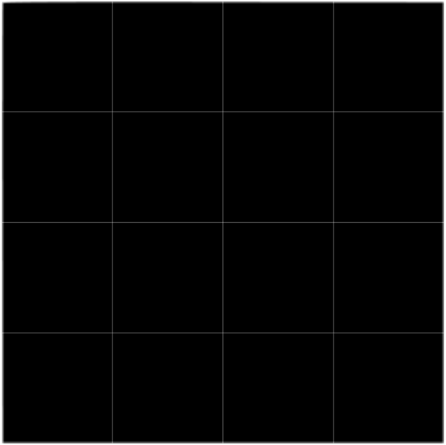

Figure: 16 squares of side length $1/4$.

Food for thought 2

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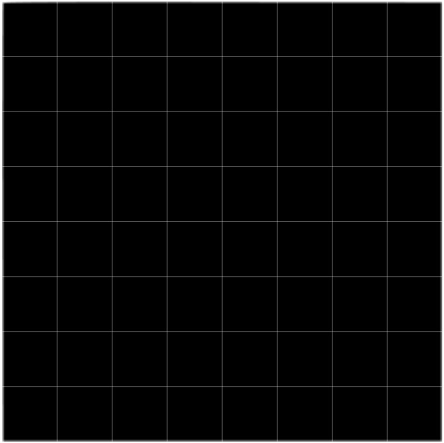

Figure: 64 squares of side length $1/8$.

Food for thought 2

Like before we start covering:

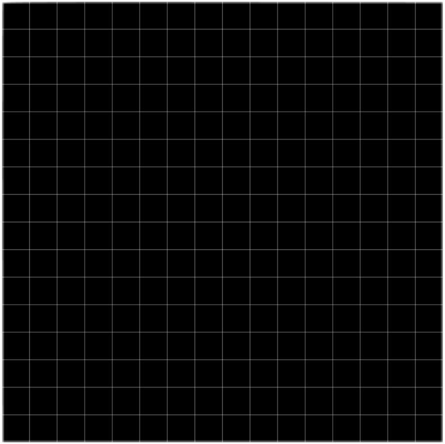

Figure: 16^2 squares of side length $1/16$.

Food for thought 2

Like before we start covering:

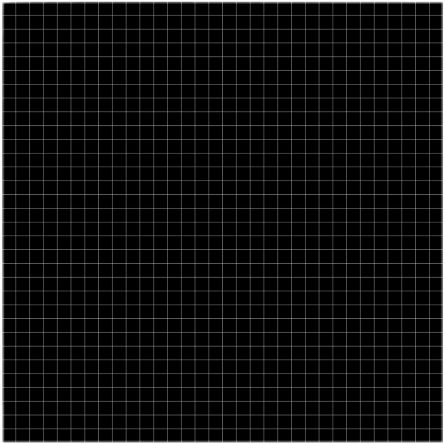

Figure: 32^2 squares of side length $1/32$.

Food for thought 2

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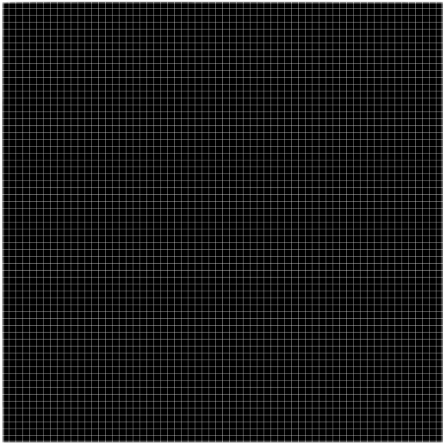

Figure: 64^2 squares of side length $1/64$.

Analyzing the numbers 2

Let's write down the numbers:

$$\begin{cases} N_{1/2} = 2^2 \\ N_{1/4} = 4^2 \\ N_{1/8} = 8^2 \\ N_{1/16} = 16^2 \\ N_{1/32} = 32^2 \\ N_{1/64} = 64^2 \end{cases}$$

It looks like whenever we halve the size of the square, the number of squares we need, multiplies by $4 = 2^2$. Again this makes total SENSE!.

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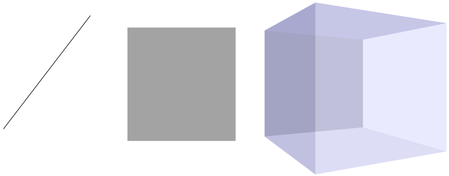
It looks like whenever we halve the size of the square, the number of squares we need, multiplies by $4 = 2^2$. Again this makes total SENSE!. Not surprisingly, to cover a square of area $\mathcal{A}$ we need:

$$\mathcal{A}\left(\frac{1}{\epsilon}\right)^{2} \text{ squares of size } \epsilon\,.$$

Intuition

This intuition works in all dimensions as well:



The growth of the covering, tells us the dimension. This is nice since we are not talking about *number of parameters*, and it could be noninteger.

We are ready for fractals now!

The first example

This is the Koch snowflake:

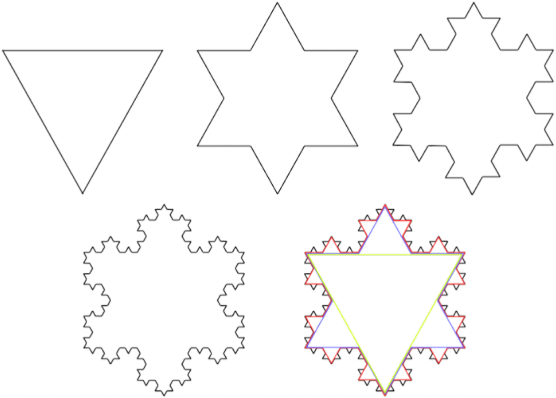


Figure: The process to make a Koch snowflake (curve).

Let's see how many boxes we need to cover it!

The covering

Let's begin covering:

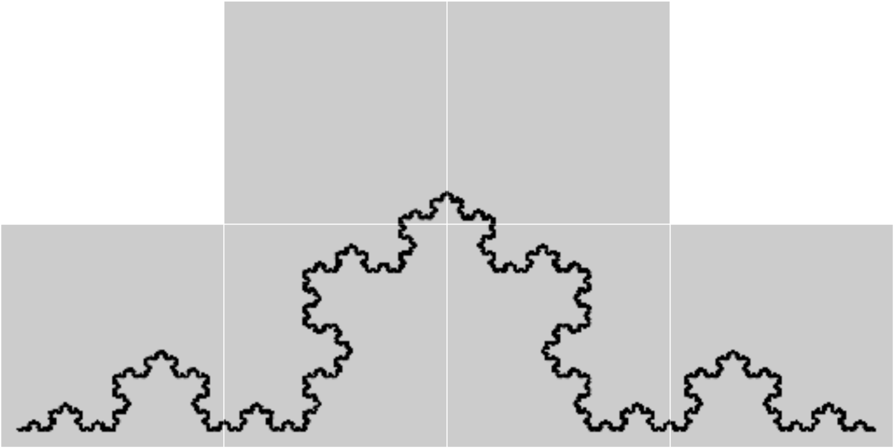


Figure: 6 boxes of length 128 pixels.

The covering

Let's begin covering:

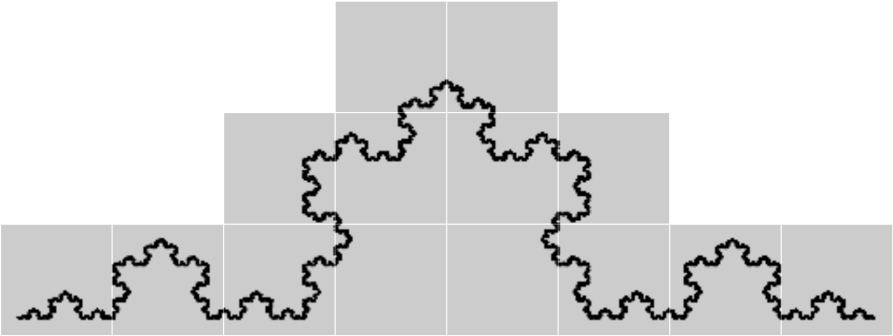


Figure: 14 boxes of length 64 pixels.

The covering

Let's begin covering:

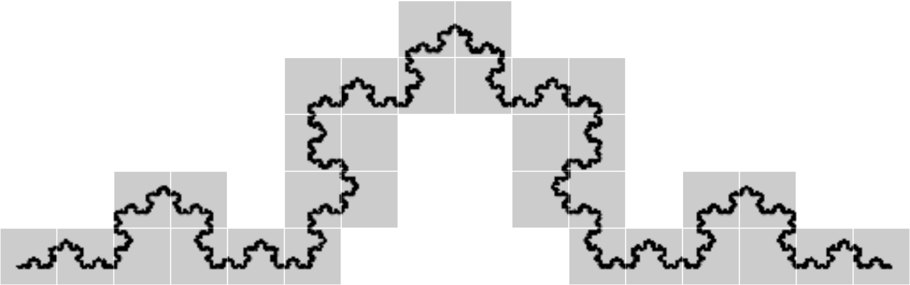


Figure: 32 boxes of length 32 pixels.

Let's begin covering:

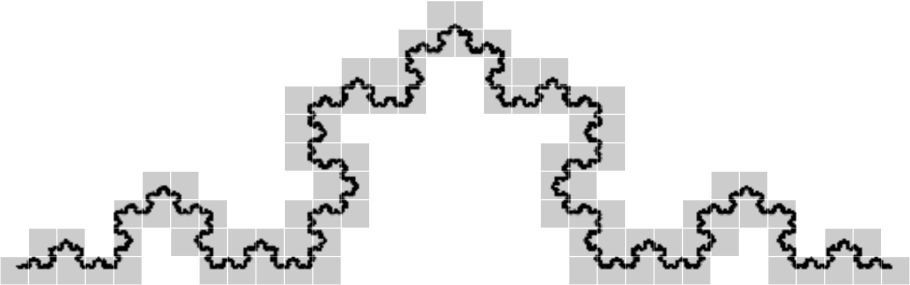


Figure: 92 boxes of length 16 pixels.

The covering

Let's begin covering:

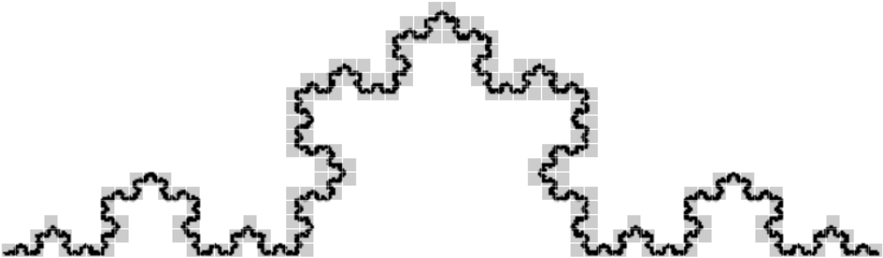


Figure: 197 boxes of length 8 pixels.

The covering

Let's begin covering:

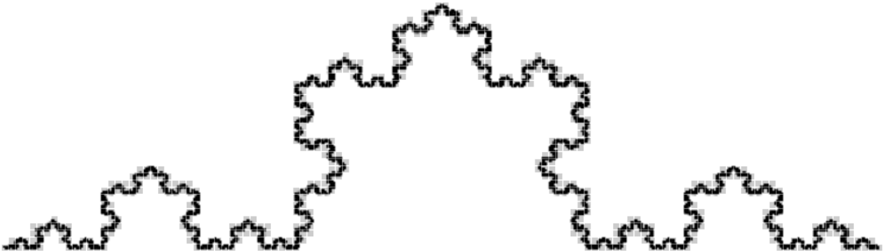


Figure: 515 boxes of length 4 pixels.

Analyzing the numbers

Let's write down the numbers:

$$\begin{cases} N_{128} = 6 \\ N_{64} = 14 \\ N_{32} = 32 \\ N_{16} = 92 \\ N_{8} = 197 \\ N_{4} = 515 \end{cases}$$

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After a little bit of analysis we see that:

$$N_\epsilon \sim C \left(\frac{1}{\epsilon}\right)^{1.26}.$$

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After a little bit of analysis we see that:

$$N_\epsilon \sim C \left(\frac{1}{\epsilon}\right)^{1.26} .$$

Everytime we halve the size of the box, the number of boxes we need multiplies by $\sim 2^{1.26}$.

The dimension of the Koch snowflake is about $1.26 \sim \frac{\log(4)}{\log(3)}$. The dimension is **fractional** and it is a **fractal**. (the origin of the word is this)

Another view

We can also measure the slope of the log–log graph:

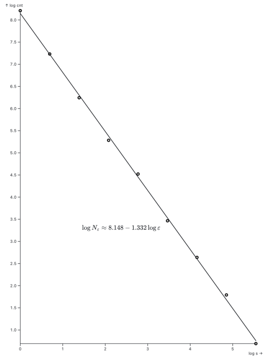

Figure: $\log(N_\epsilon)$ plotted against $\log(\epsilon)$

This is the dimension, (might have errors because of the resolution but it's close $1.33 \sim 1.26$)

The box-counting dimension

This is in fact the definition of dimension. It is very simple, beautiful and at the same time very powerful (Both practically and theoretically).

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The rigorous definition is the following:

Minkowski–Bouligand (box–counting) dimension

The dimension of any set $S \in \mathbb{R}^n$ is the following limit:

$$\dim_{\text{box}}(S) = \lim_{\epsilon \to 0} \frac{\log(N_\epsilon)}{\log(1/\epsilon)}\,.$$

Here N_ϵ is the minimum number of boxes of size ϵ we need to cover S.

Now let's do some more experiments.

The tree

Let's find out the dimension of the tree we saw in the beginning:

Figure: 51 boxes of size 128.

Making boxes smaller:

Figure: 196 boxes of size 64.

Making boxes smaller:

Figure: 702 boxes of size 32.

Making boxes smaller:

Figure: 2234 boxes of size 16.

Making boxes smaller:

Figure: 5853 boxes of size 8.

The tree

Making boxes smaller:

Figure: 13627 boxes of size 4.

Analyzing

Let's write down the numbers:

$$\begin{cases} N_{128} = 51 \\ N_{64} = 196 \\ N_{32} = 702 \\ N_{16} = 2234 \\ N_{8} = 5853 \\ N_{4} = 13627. \end{cases}$$

After a little analysis, we see that every time the box halves in size, the number of boxes multiplies by $\sim 2^{1.7}$. The dimension of this tree is 1.7.

Example: leaf

For the leaf we cover as follows:

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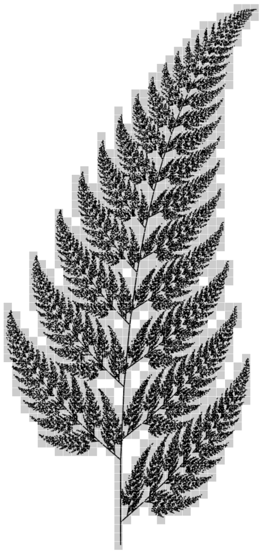

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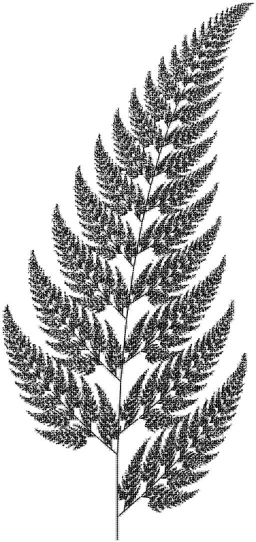

This leaf has 1.74 dimension:

The box-count data for this set is:

s	1	2	4	8	16	32	64	128	256	512
N_s	97590	37834	12378	3609	1028	294	87	27	8	2

yielding a fractal dimension estimate of approximately 1.744. The regression line is

$$\log N_s \approx 11.719 - 1.744 \log s,$$

Hausdorff measure

We can also find the s dimensional volume of a set $S \subset \mathbb{R}^2$. For a curve it's easy:

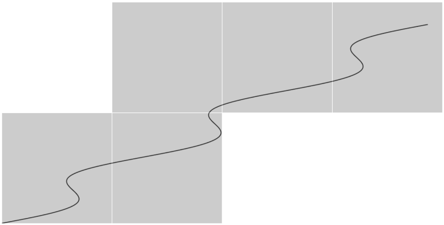

Figure: 5 boxes of size 256 pixels.

We sum the number of squares time the side length:

$$5 * 256 = 1280\,.$$

Length of curve

Making finer estimates:

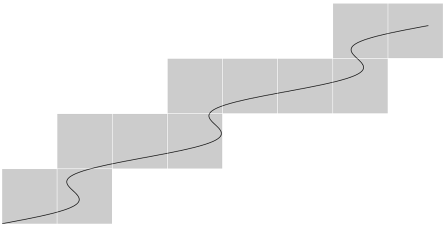

Figure: 11 boxes of size 128 pixels.

We sum the number of squares time the side length:

$$11 * 128 = 1408\,.$$

Length of curve

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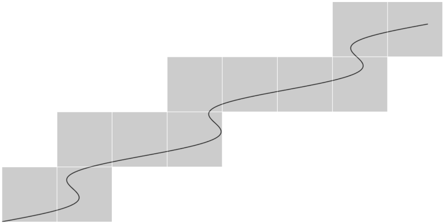

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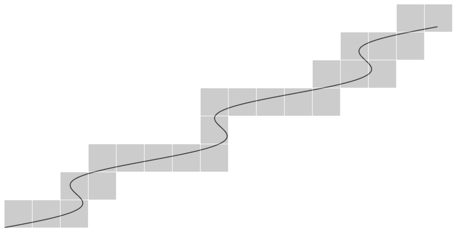

Figure: 24 boxes of size 64 pixels.

We sum the number of squares time the side length:

$$24 * 64 = 1536 \,.$$

Length of curve

Making finer estimates:

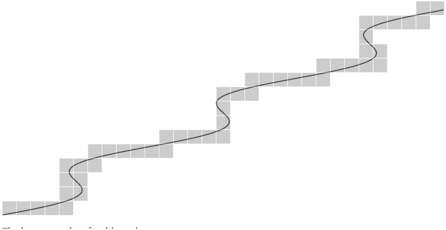

Figure: 49 boxes of size 32 pixels.

We sum the number of squares time the side length:

$$49 * 32 = 1568\,.$$

Length of curve

Making finer estimates:

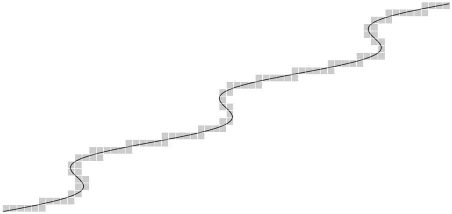

Figure: 101 boxes of size 16 pixels.

We sum the number of squares time the side length:

$$101 * 16 = 1616 .$$

Length of curve

Making finer estimates:

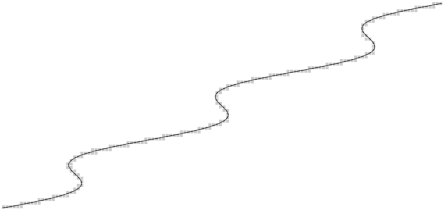

Figure: 201 boxes of size 8 pixels.

We sum the number of squares time the side length:

$$201 * 8 = 1608\,.$$

And this is the best estimate.

Haussdorf measure

Remember the Koch snowflake had dimension $\log(4)/\log(3)$. Let's find the $\frac{\log(4)}{\log(3)}$-dimensional volume of the Koch snowflake:

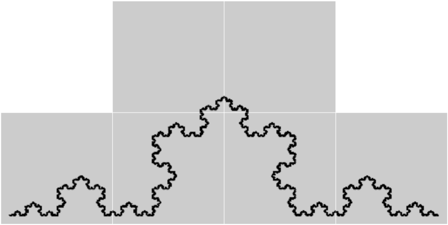

Figure: 6 boxes of length 128 pixels.

We calculate:

$$6 * 128^{\frac{\log(4)}{\log(3)}} \sim 2725\,.$$

Haussdorf measure

Remember the Koch snowflake had dimension $\log(4)/\log(3)$. Let's find the $\frac{\log(4)}{\log(3)}$-dimensional volume of the Koch snowflake:

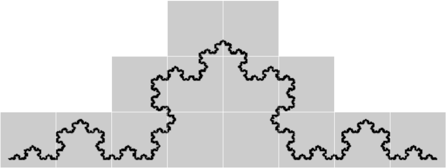

Figure: 14 boxes of length 64 pixels.

We calculate:

$$14 * 64^{\frac{\log(4)}{\log(3)}} \sim 2653\,.$$

Haussdorf measure

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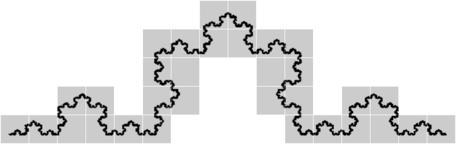

Figure: 32 boxes of length 32 pixels.

We calculate:

$$32 * 32^{\frac{\log(4)}{\log(3)}} \sim 2530\,.$$

Haussdorf measure

Remember the Koch snowflake had dimension $\log(4)/\log(3)$. Let's find the $\frac{\log(4)}{\log(3)}$-dimensional volume of the Koch snowflake:

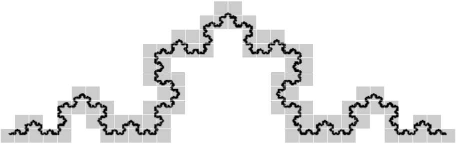

Figure: 92 boxes of length 16 pixels.

We calculate:

$$92 * 16^{\frac{\log(4)}{\log(3)}} \sim 3035\,.$$

Haussdorf measure

Remember the Koch snowflake had dimension $\log(4)/\log(3)$. Let's find the $\frac{\log(4)}{\log(3)}$-dimensional volume of the Koch snowflake:

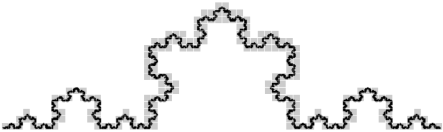

Figure: 197 boxes of length 8 pixels.

We calculate:

$$197 * 8^{\frac{\log(4)}{\log(3)}} \sim 2712\,.$$

Haussdorf measure

Remember the Koch snowflake had dimension $\log(4)/\log(3)$. Let's find the $\frac{\log(4)}{\log(3)}$-dimensional volume of the Koch snowflake:

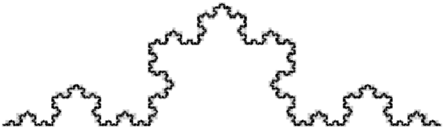

Figure: 515 boxes of length 4 pixels.

We calculate:

$$515 * 4^{\frac{\log(4)}{\log(3)}} \sim 2958\,.$$

This is the best estimate.

Epilogue

There are many different definitions of dimension with their own special properties:

- Hausdorff, Minkowski, Assouad and many more.
- Some are bigger than the others, some are more useful in certain situations and easier to study given certain tools.

However they share a simple fact: To determine the dimension of a set, we have to look at finer and finer scales (zoom in).

A famous problem: Kakeya sets

Let me talk about an old, famous and beautiful problem in this field:

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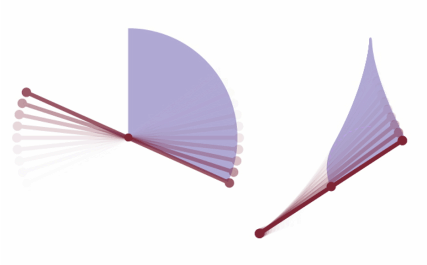
- Imagine you have a needle of unit length in the plane. Anywhere you move the needle, it colors its trace:



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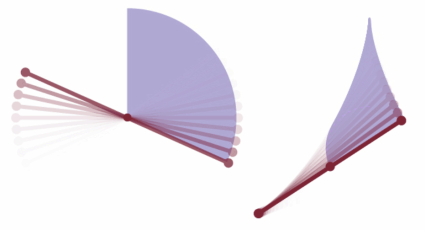
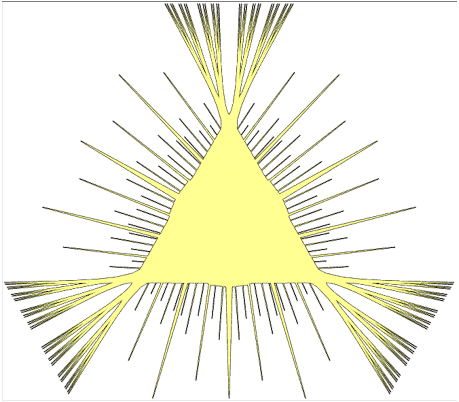


- The goal is to turn the needle 180 degrees, while coloring the least area possible. How much is this area?

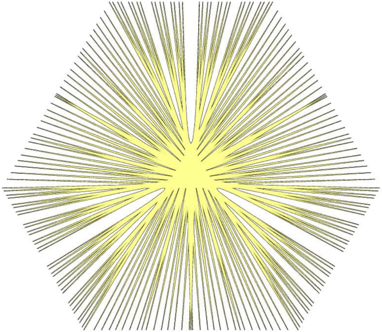
Kakeya: continued

The answer is 0! CRAZY, right? In fact for any small $\epsilon > 0$, there is a way you can turn over the needle with the area traced less than ϵ.

The answer is 0! CRAZY, right? In fact for any small $\epsilon > 0$, there is a way you can turn over the needle with the area traced less than ϵ. The smaller you make $\epsilon > 0$, the pointier the set become:



We can make ϵ smaller and smaller and take a *limit* of these crazy pointy looking sets.



Kakeya sets

We define:

Kakeya sets

It's a set $S \in \mathbb{R}^n$ that has unit segments in *every direction*.

Kakeya sets

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These sets can be super small, What about their dimension? In 1917 Kakeya proposed the following conjecture

Kakeya Conjecture

Any Kakeya set in $\mathbb{R}^n$ has (Hausdorff) dimension at least n.

Resolutions: dim 2,3

In dimension 2 this problem has been solved for a long time, however the case of dimension 3 remained unsolved for more than 50 years until last month! Our own **Hong Wang** along with **Joshua Zahl** cracked the problem (in a 127 page paper).
Many have cited this progress as one of the most exciting and important of this century.
Today, we could understand what this question even means.

Minimal surfaces

Bubbles, always find the minimum area possible! either with a fixed boundary:

Figure: Bubbles in Central Park (J.A.)

Or they minimize area with some air trapped inside:

They have also places that than two bubble sheets touch, with a different angle than 180. These places are called *Singular points*. One can ask how big is this set? Or we can ask

How big is the **dimension** of the Singular set?

There are still fundamental facts that are unknown about this problem.

Bubbles

They can also look more complicated, or be very unstable:

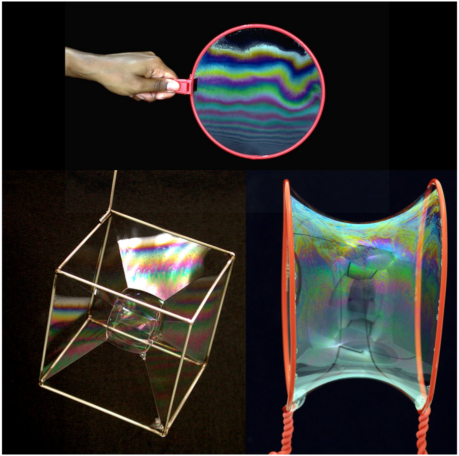
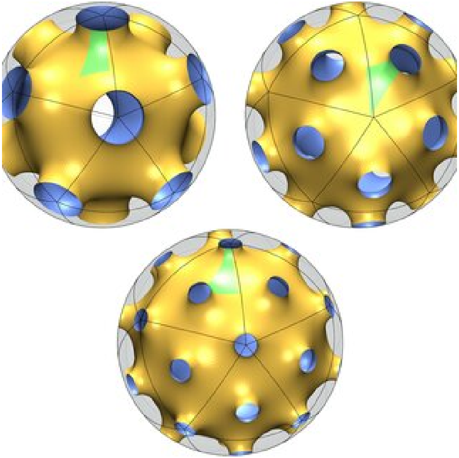


Figure: Minimal surfaces

Minimal surfaces

What do they look like? Can we say anything about them?



Finale

And I hope after this talk, you can look a little differently at the nature around you.
and maybe ask a bit more...

- What about the dimension of your lungs?
- what about the dimension of your neurons in your brain?
- What about the dimension of the roads in a city? What about the dimensions of a Broccoli?

Today's talk was a topic in the vast field of **Geometric Measure Theory**.
In GMT we study and analyze geometric shapes and their properties.

Thank you for your attention!